AUTHOR'S CONTRIBUTION

by Assoc. Prof. Danail Brezov

Description of the publication (in the order, presented for the application):

[1] This chapter considers low-dimensional geometric models based on rational numbers and constructed as homogeneous spaces of Clifford algebras with integer coefficients. The connection with Gaussian (respectively quaternion) primes, Pythagorean spinors and discrete Hopf bundles is traced. Special attention is paid to the motion decomposition task in this construction, taking into account the difference between the Euclidean and hyperbolic metric cases. The groups of motion are constructed with dual expansion, the possibility of generalization in higher dimensions is commented on, as well as the construction of quasi-differential geometry on these objects, using the apparatus of non-standard analysis. Examples from physics are also proposed: the fractional-linear decomposition for the one-dimensional quantum scattering problem and the rational parametrization for the so-called 'Wigner little groups' of $SO^+(3,1)$.

[2] Based on a previously obtained necessary and sufficient condition for the decomposability of three-dimensional rotations with generalized Euler angles, the paper investigates the question of the finite generation of elements in SO(3) given two or more generators in the form of points on the unit sphere. The solution to this problem boils down to an estimate of the length of the geodesic path connecting these points, which generalizes Lowenthal's classical result for decomposition with only two generators. The possibilities for optimization that the larger number of factors provides are also commented on and the non-homogeneous case where the generalization is given directly by the transfer principle is discussed as well.

[3]The paper mainly focuses on statistical data processing, construction and training of machine learning models aimed at filling missing information in the metropolitan traffic map, pollution with nitrogen oxides and other key indicators for the air quality in Sofia city. To overcome the deficit of reliable data, we apply collective regression models (Random Forest, Extreme Gradient Boosting etc.) in two stages for the primary and secondary street network, respectively. The estimated accuracy of the predicted results is further increased with the use of modern AutoML libraries in Python, which combine several models into a weighted sum (level 2 stacking), and thus improve the efficiency significantly.

[4] Like the previous publication, this one is also part of a collective effort on a research project 'Developing a methodology for assessing air quality and its impact on human health in an urban environment' under the National Institute of Scientific Research. Unlike that, however, I am not the leading author here and my contribution is more modest: in addition to the regression model describing the concentration of nitrogen oxides in the urban environment, I have also helped to ensure representativeness in the sociological study accompanying the statistical assessment of long-term effects. We have investigated the impact of urban noise and pollution, as well as access to green spaces, on human health according to the survey.

[5] The realization of the Lorentz group as a complex projective line with an additional structure (quaternion multiplication) allows for a number of analogies to be made between familiar phenomena from classical Newtonian mechanics and exotic effects in Special Relativity. Here I take advantage of this opportunity by investigating the 'complex kinematics' of pseudo-rotations in $\mathbb{R}^{3,1}$ in relation to electromagnetic fields in vacuum. A prime example is the well-known Coriolis effect, interpreted as a (classical) geometric phase, and in the complex realization is associated with the Thomas precession, the Sagnac effect and the Hall effect, diligently explored in various areas of light theory and elementary particles. The connection with the Hopf bundles and their non-compact counterparts, the importance for quantum computing, and the possibility for a physical interpretation of the complex angular momentum are here commented as well.

[6] Here we propose a solution to a practical question: how does a squadron of pilots (or robot drones) make formations in the absence of central navigation or a reference object for localization in space? The two-dimensional case (for example, a tank battalion in the desert) is significantly simpler and is easily implemented with complex numbers presented in trigonometric form, and for non-commutative three-dimensional rotations it turns out to be convenient to use projective quaternions due to the smaller number of parameters. Applying some symmetry considerations, we reduce the solution to intersecting quadrics, with the compatibility conditions of the system depending on the viewing angle of the cameras and their initial orientation. We also make an estimate for the average number of objects that each camera can see, and at the end we comment on the corresponding kinematic task (the continuous case).

[7] The article examines the hyper-complex algebras arising from the iterated vector product in $\mathbb{C}^3$, as well as the analytic properties of functions acting on them (generalized Cauchy-Riemann conditions). The complex numbers themselves arise from the restriction of this construction to the real case, but it also contains some more exotic representatives, such as para-complex, bi-complex, and dual algebras, among lesser-known examples. In higher dimensions, instead of a vector product, we use an external multiplication coupled with the Hodge operator. The construction is applied to 'algebraize' the Frenet (Cartan) moving frame of a smooth parameterized curve.

[8] This paper introduces a complex analogue of the Rodrigues rotation formula for orthogonal operators in $\mathbb{C}^3[\varepsilon]$. With its help, an algorithm for powers and roots of such operators is proposed, generalizing De Moivre's construction in the complex plane. Combined with polar decomposition, the method gives a solution for arbitrary operators, and in the nilpotent case it is surprisingly simple. The decomposition $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$ also extends the result for linear operators in $\mathbb{C}^4[\varepsilon]$, as in higher dimensions for decomposable elements of the group, the Plücker embedding provides an efficient reduction that can be easily described with techniques borrowed from algebraic geometry, as I show in an earlier paper. This reduction preserves standard structures from the three-dimensional case, such as the generalized Euler angle decomposition for example.

[9] The work is devoted to the projective quaternion technique and its applications in the description of low-dimensional motion groups - from the classical case of rotation with a fixed point, to the Galilean and Lorentz groups. The focus is on the projective description of kinematics (Maurer-Cartan style) and the curious geometric correspondence between curves in $\mathbb{CP}^3$ (the group manifold) and $\mathbb{C}^3$, where the generalized angular velocity takes values. The dual case is discussed as well, and so are the dynamic aspects of our construction. Concrete examples are shown with an emphasis on applications in physics.

[10] With the aid of quaternion techniques, we obtain the solutions of a broad class of kinematic problems in the form of SU(2)-propagator and projective $\mathbb{RP}^3$ vector-parameter. At first, all motions with a fixed direction of angular velocity (or the magnetic field in the quantum-mechanical analogue) are systematized. We then describe rotations with a precessing angular velocity, which in spin systems corresponds to the Rabi oscillator used in nuclear magnetic resonance and certain types of quantum computers. Finally, using a standard technique known as the 'interaction picture', we obtain the solutions in the general case, with a simple integrability condition generalizing the Rabi resonance, which in this context we interpret as an algebraic propagator decomposability condition. If the latter are not met, we use instead an iterative procedure based on the famous Magnus expansion.

[11] We present an alternative parametrization of three-dimensional rotations based on decomposition into pairs of factors. The invariant axis of the first is fixed, while the second one precesses about it in a suitable manner so as to satisfy the decomposability condition. The two axes remain at all times mutually perpendicular, which greatly simplifies the calculations. The problem also has a hyperbolic analogue with applications in scattering theory, which is discussed in another publication. The focus here is on rigid body mechanics: expressions for the angular momentum operator and the Laplacian in this parametrization are derived, as well as the kinematic and dynamic equations. We also find a class of analytic solutions, discuss various possibilities for implementations and possible generalizations.

[12] Covariant solutions to the problem of decomposition of three-dimensional rotations into generalized Euler angles (with respect to arbitrary axes) are proposed in the paper. The solutions are expressed by the components of the vector-parameter (the Rodrigues vector) for the composite rotation in the basis formed by the invariant axes, and in the coplanar case this basis is supplemented by the normal to the plane they determine. Fractional linear expressions are used for the covariant and contravariant components in the quasi-basis defining the problem, singular solutions and infinite points are studied too.

Other publications

In the articles related to my dissertation, I focus on the so-called 'vectorial parametrization' using a projectization of the corresponding Clifford group (quaternion or pseudo-quaternion) in both the Euclidean and hyperbolic settings. The low-dimensional cases of Euclidean and Lorentzian metrics are obtained without much effort, while in higher dimensions (but still within the set of Hurwitz algebras) formal fractional-linear functions with the so-called 'Vahlen matrices' are used. One of the contributions of the later papers was to formalize this method and place it in the context of geometric algebras. This allows some freedom of expression: for example, in the description of the Lorentz group, we have at our disposal a rational representation with a unified Rodrigues (in a broader sense, Cayley) formula for all types of transformations, including the non-orthochronous ones that occur naturally as an analytic continuation. The description of the Möbius group of the real line and the classical hyperbolic geometry in the Poincaré model also emerges naturally with this approach. Another central result in my older publications is obtaining analytic solutions for the generalized Euler decomposition in $\mathrm{SO}(3)$ and $\mathrm{SO}(2,1)$. Previous attempts propose instead of explicit formulas, a complex algorithm, which does not always work because of the inevitable singularity (known to engineers as the 'gimbal lock problem') any smooth map between the projective space and the torus is bound to have. In the non-compact setting we have also an isotropic singularity associated with the more exotic features of vectors on the light cone. It wroth investigating the necessary and sufficient conditions for decomposability in both cases: for rotations it reduces to definiteness of an expression that we can interpret as the Gram determinant for the moving (attached to the rigid body) system of axes, while for hyperbolic pseudo-rotations the isotropic singularity complicates the issue, but there one may resort on the Iwasawa decomposition. In both cases, however, the optimal parameterization yields extremely elegant and practical solutions.

Mathematical Analysis (Second Part)

The textbook includes basic concepts from the theory of ordinary differential equations, integration on manifolds and vector calculus, as well as the differential geometry of curves and surfaces. It adopts a practical approach suitable for engineering students - new ideas are introduced with examples, theorems are not always rigorously proven, while in return geometric and mechanical interpretation is often provided. In ODE's, basic techniques for solving elementary problems are first mastered, and only then the existence and uniqueness theorem is discussed along with Picard's method. In the case of higher-order equations, the representation with first-order vector ODE's is also used in parallel, in order to build a geometric intuition for the properties of the flow, the fundamental system of solutions, the spectrum of the problem and the phase portrait. Covered (in a simplified version, with more examples) are topics such as numerical and iterative methods for finding approximate solutions, Taylor series expansion and the Frobenius method, operational calculus with the Laplace transform, linearization and qualitative analysis, perturbations and chaos theory. Interesting examples from physics, biology, population dynamics and economics are offered, along with plenty of illustrations. The section on differential geometry begins with the introduction of the tangent line and the curve integral as a tool for calculating distance and work. Following physical intuition, we introduce the moving Cartan frame and derive the Frenet-Serret equations defining the curvature and torsion. Surfaces follow this sequence: understand the concepts such as normal, gradient, tangent plane, and surface vector (with examples), learn how to calculate areas and flows, and only then define Gaussian and mean curvature. The theorems of Green, Stokes and Gauss are introduced via differential forms, as is most natural, and then interpreted with the ∇-formalism. Due attention is paid to standard coordinate changes and their Jacobians. The theory of potential and its connection to exact ODE's is commented on, and the variational approach is briefly introduced at the end of the textbook, with two interesting examples: geodesic and minimal surfaces.