REVIEW

from ass. prof. Dimitar Vladislavov Atanasov, PhD
dep. Informatics, New Bulgarian University,
4.5 Mathematics
for obtaining the scientific degree of **Doctor** in a professional field
4.6. Informatics and computer science
with candidate **Neviana Dimitrova Georgieva**

The review is in accordance with Order 3-RK-233 of 05.05.2022 of the Rector of New Bulgarian University, appointing a scientific jury for the doctoral dissertation defense of Nevyana Dimitrova Georgieva, full-time doctoral student with a scholarship in the doctoral program in Informatics.

The materials provided to the protection procedure include:

- Dissertation thesis;
- Abstract;
- Autobiography;
- Copy of order for appointment of the scientific jury;
- Enrollment order;
- Deduction order;
- Transcript of the agenda of the Faculty Council of the Master's Faculty of NBU for final attestation of the doctoral student.

Nevyana Georgieva is a full-time doctoral student at the Department "Informatics New Bulgarian University, enrolled by order N61 / 26.10.2012, deducted with the right to defensed by order of the Rector of NBU 3-RK-350 from 19.07.2017. During the period of study, the doctoral student used a break of 1 year.

The dissertation, entitled "Network Coding and Analogs of Design", examines the problem of the existence of designs in geometries with projective coordinates over finite rings. The study of such problems began in the early twentieth century with the work of Hjelmslev. In the second half of the century there was significant progress in research in this area.

The motivation for this dissertation comes from coding theory, but in essence it can be related to combinatorial configuration theory (designs) or to the finite projective geometries. The main task of the dissertation is to find the necessary and sufficient conditions for the existence of a special structure called a spread in some little-known geometries - the coordinate geometries above the finite chain rings. Spread is a set of subspaces of a certain type that cover all points of geometry. The connection with coding theory is due to the fact that such sets can be used as codes in a special metric - the rank metric - and be used to transmit data in a new type of coding - network coding.

The issue of finding the necessary and sufficient conditions for the existence of spreads from fixed-dimension subspaces has been resolved in the case of the projective Galois geometries $PG(r,q)$. There it turns out that the obvious necessary condition, namely, the number of points in the subspace to divide the number of all points is sufficient as well. In Hjelmslev's geometries, this is true only for the case of spreads from Hjelmslev subspaces, ie. subspaces associated with modules. In general, this condition is not enough and the solution to this issue is much more difficult. In the present dissertation the solution of this problem in some important special cases is presented. Examples have also been constructed in which spreads of subspaces of given (non-free) types do not exist. The more general task for the existence of the so-called R-analogs of designs in projective geometries of Hjelmslev.

The dissertation has a volume of 78 typewritten pages and consists of four chapters and a list of references, including 85 titles. Here is a summary of the content of the dissertation and the results obtained in it.

Chapter 1 is introductory. It begins with a brief historical overview of the development of those areas that are relevant to the dissertation work. First of all, these are the coordinate geometries above the rings, which date back to the early twentieth century. Initially, these were extensions of real numbers, such as Study's dual numbers, and later geometries over random chain rings were considered. In the 1970s and 1980s, a study of codes began regarding the so-called rank metrics, and even later in 2008 the foundations of network coding were laid with a work by Koetter and Kschischang. A network code is, roughly speaking, a set of subspaces in some geometry that have some metric properties with respect to the rank metric. In the present work, subspaces are taken from coordinate geometry over a chain ring. Further, the chapter represents an overview of the obtained results.

Chapter 2 is devoted to the presentation of the main theoretical results used in the dissertation. It is structured in three parts, in which results are presented sequentially for chain rings, for modules above chain rings and for design and affine coordinate geometries above chain rings. The first section presents a result describing the general structure of an finite chain ring. In general, finite chain rings are factor rings on Galois rings; they are not necessarily commutative and are characterized by a degree of prime number. This section also defines a linear regulation of the elements of a chain ring, which is used later when entering the standard matrix shape. The complete classification of chain rings with nilpotency index 2 is also described.

Section 2.2, which deals with modules over finite chain rings, contains two important results: the first of them (Theorem 2.7) is a structural theorem, according to which each finite-generated module above a chain ring is a direct sum of cyclic modules. The other important result is Theorem 2.8, which derives exact formula for the number of submodules of type μ contained in another submodule of type λ. This formula can be considered as a generaliation of Gaussian coefficients.

Section 2.3 presents some of the more important structural results for coordinate geometries above finite chain rings, which are Hjelmslev geometries. These geometries can also be introduced axiomatically as in the case of Desargues together with some additional conditions (sufficient number of points on lines) Hjelmslev's geometries are exactly the coordinate geometries above chain rings. Yelmlsev's geometries have an interesting nested structure, which, although more complex than that of the Galois geometries, allows for effective research. Several theorems (2.11-2.14) present important results such as that the factor structure of Hjelmslev geometry is again Hjelmslev geometry over a chain ring with a lower nilpotency index.

The original results for the dissertation are contained in Chapters 3 and 4.

Chapter 3 is devoted to a problem which, although secondary, is important for any research in Hjelmslev geometries and in general when working with modules over finite chain rings. This is the question of presenting a module. Of course, each module can be set with its basis, but this representation is not unique. For example, the question of whether rows of two different matrices generate the same module is not obvious. This chapter introduces a special form of matrix, which is unique for each module, whose rows generate the module and which allows an easy comparison and easy operations.

This representation is introduced by Definition 3.1.1 and is analogous to the so-called row-reduced echelon form of the matrix. The most important result here is Theorem 3.3, which states that with each module above the chain ring R a single matrix in a standard form can be connected, the rows of which generate the module. From this result it can be proved that each module is isomorphic to a module with generating matrix in a triangular (block) shape. This corresponds to the standard generating matrix of linear codes above finite fields. This is described in Corollary 3.4. From this standard form, an explicit generating matrix for the orthogonal module $M_R^\perp$ of the output module ${}_RM$ can be obtained.

Later in this chapter, the introduced standard form is used for easier operation of modules. Several algorithms for the following tasks are presented in pseudocode: obtaining a matrix in a standard form, finding the union and section of modules, checking the membership of a vector to a module, finding the orthogonal of a module, generating all submodules, generating all submodules of a given type contained in a fixed module.

Chapter 4 is dedicated to the so-called R - analogues of designs and in particular of spreads. The latter are defined as sets of subspaces in a given Hjelmslev geometry, which represent a breakdown of the point set of geometry. This chapter is also divided into three parts.

Section 4.1 introduces R-analogues of designs in a similar way to q-analogues of designs above finite fields. This is a family of fixed-type submodules having certain regular properties: for example, each sub-module of a given type must contain a fixed number of times or at least a fixed number of times. Some simple combinatorial conditions necessary for the existence of such designs are proved and the connections between the different types of designs are described. Particularly important, as in the classic case, are designs in which each submodule occurs exactly once. By analogy with the classical case, they are called Steiner systems. The spreads that are studied in the next section are exactly Steiner systems. There are no other known Steiner systems for both q-analogs and R -analog designs.

Section 4.2 contains results on the existence of spreads. In particular, it demonstrates one necessary and several sufficient conditions. For spreads from free subspaces (the so-called Hjelmslev subspaces) the problem was considered by Kiermaier and Landjev. It turns out that the combinatorial necessary condition is sufficient, just as in the Galois geometries $PG(r, q)$.

The more important results in this section are Theorem 4.8, where a necessary condition for the existence of spreads is given, which generalizes that of Kiermaier-Landjev (Theorem 4.7). Next, several sufficient conditions for the existence of spreads of non-free subspaces are proved (Theorems 4.10–1.12). Finally, a hypothesis is presented as a necessary and sufficient condition for the existence of spreads in Hjelmslev's geometries, for which there is currently no counterexample. The whole section 4.3 is devoted to the construction of an example to demonstrate that the trivial combinatorial condition (ie the number of points in subspace of the spread to divide the number of points in the geometry), which is necessary is not always sufficient. In the only theorem here (Theorem 4.13) it is proved that in spaces above rings R with nilpotency index 2 for all even $n \geq 4$ in the geometry $PHG({}_RR^n)$ there are no spreads from subspaces of type $\lambda = 2^{n/2}1^a$, where $1 \leq a \leq \frac{n}{2}$. This gives a wide class of types of subspaces for which the combinatorial necessary condition for the existence of spreads is not sufficient. To date, the necessary and sufficient condition for the existence of spreads of (non-free) subspaces in Hjelmslev's geometries has not been obtained, although the present work is an essential step in this direction.

From the exposition made so far it can be summarized that the main scientific contributions of the dissertation are:

- A standard form is found for matrices above an end chain ring, in which for each finitely generated module there is a single matrix in standard form, whose rows generate the module.
- For a right module generated by a matrix in a standard form, a matrix was found that generates the orthogonal module.
- The results described above are illustrated by algorithms for modules.
- Sufficient conditions have been established for the existence of spreads of non-free modules
- Examples of λ types are shown for which the required combinatorial condition is not sufficient.

The author used 85 literature sources in the thesis. The earliest of them are from the first half of the twentieth century, when the main problems in the field were raised. The predominant part of the cited literature is from the last 20 years, which shows that the PhD student has a view both on the classic works in the field and on the current results published in recent years. The thesis shows in-depth knowledge and experience in the subject area of the dissertation. The main results described in the paper are published in three publications, one of which is independent and two in co-authorship with the supervisor. One of them is in a prestigious journal from Q2 on WoS. Two of the publications are in approwed Bulgarian journals.

The main results of the thesis are reported at 8 scientific forums, 7 of which are international. Three of the presentations are at the scientific conference Computer Science and Education in Computer Science, which is traditionally organized by NBU, together with the universities of Fulda (Germany) and Boston (USA). The PhD student has delivered a talk four times at the international forum Workshop on Algebraic and Combinatorial Coding Theory.

It can be concluded that the results of the dissertation have been sufficiently approved and have gained popularity both in the country and abroad.

Personally and professionally, I have known Nevyana Georgieva since her student years, when she was working on her master's thesis in the field of probability theory and branching stochastic processes. Subsequently, she became a full-time lecturer at the Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski"and started working in the field of coding theory. Since 2020 she is a part-time lecturer at the Informatics Department of NBU, where she teaches mathematical disciplines using specialized software. The high opinion of the students about her teaching qualities and diligence in the work process should be teken in account.

Conclusion. In view of the above results, I confidently give my **positive assessment** for the dissertation, achievements and contributions, I propose to the esteemed scientific jury to award the educational and scientific degree "Doctor"to Nevyana Dimitrova Georgieva in the field of higher education 4. Natural sciences, mathematics and informatics, professional field 4.6. Informatics and computer science.

ass. prof. D. Atanasov, PhD